Webology (ISSN: 1735-188X)
Volume 18, Number 4, 2021

# Analysis The Rings Of Commutative In Idempotent

**Amarendra Kumar Pattanayak and Dr. Arihant Jain**

Department of Mathematics, Dr. A. P. J. Abdul Kalam University, Indore (M.P.), India

Corresponding Author Amarendra Kumar Pattanayak

---

**ABSTRACT**

Given a commutative ring R and its quotient ring R/N, where N is often a nilpotent ideal of R, a formula is given for computing the idempotent elements of R. This formula is used to characterise idempotent elements in particular commutative rings. In order to better illustrate the key findings, several instances have been provided.

**Keywords:** nilpotent ideal, Idempotent element, group algebra, commutative ring, chain ring

## 1. INTRODUCTION

An idempotent element, also known as an idempotent of a ring, is an element of the ring a such that $a^2 = a$, in the context of ring theory, a subfield of abstract algebra. [1] The element exhibits idempotence under the multiplication of the ring. For any positive integer n, it follows inductively that $a = a^2 = a^3 = a^4 = ... = a^n$ .To give just one example, a matrix ring's idempotent element is a matrix that is idempotent in itself[3].

Decompositions of modules and the ring's homological features are linked for rings in general, when members are idempotent under multiplication. Boolean algebra is all about rings where adding and multiplying have no effect on the parts that make up the ring[2].

Rukhsan Ul Haq (2017) The spectral decomposition of the wavefunction and the hermitian operators(observables) in terms of the spectral projections makes projection operators crucial to the algebraic formulation of quantum theory. Hermitian operators, including projection operators, are idempotent. They are referred to as "quantum idempotents" by our group. Projection operators are significant for our mental grasp of quantum theory as they also reflect the observation process on a quantum system. This paper delves into the algebra of quantum idempotents and demonstrates that it gives rise to a number of other interesting algebras, including the Lie algebra, the Grassmann algebra, and the Clifford algebra, all of which are known as geometric algebras because they were originally developed for the geometry of spaces. This novel understanding of these geometric algebras as the underlying algebras of quantum processes and as a bridge between geometry and quantum theory is made possible by the projection operator representation. It is important to remember that projection operators span both projective geometry and quantum logic lattices. The algebras of framed braid groups, parafermions, and quarks (su(3)) will all be represented in an iterative fashion. As iterant algebra encodes both the spatial and temporal characteristics of recursive processes, these representations are stunning. This novel way of looking at fermions, spins, and parafermions is made possible by our representation of these algebras in physics (anyons) .[5]

## 2. IDEMPOTENT RINGS AND THEIR VARIETIES

The following is a partial list of significant classes of idempotents:[4]

If (ab) = (ba) = 0, then (a) and (b) are orthogonal idempotents. Since a and b are orthogonal, if an is idempotent in the unitary ring R, then so is b = 1 a.

When axe = xa for all x in R, we say that an is central to R and that an is idempotent in R.

The elements zero and one are always idempotent, hence they qualify as "trivial idempotents."

A non-zero idempotent such that aR is not a direct sum of two non-zero submodules is said to be a primitive idempotent of ring R. To put it another way, if the expression a = e + f cannot be written in the form a = e + f, where e and f are nonzero orthogonal idempotents in R, then an is a primitive idempotent.

To put it another way, an idempotent such that aRa is a local ring is called a local idempotent. This means that local idempotents are also primitive, as it follows that aR is directly indecomposable.

In mathematics, an idempotent a for which aR is a simple module is said to be irreducible on the right. Right (and left) irreducible idempotents are local since $End_R(aR)$ = aRa is a division ring and, by Schur's lemma, a local ring.

We say that an idempotent a is centrally primitive if it can't be written as the product of two non-zero orthogonal idempotents.

If there exists an idempotent b in R such that b + I = a + I, then the ring R/I has an idempotent a + I, and the ring R/I lifts modulo I.

If RaR = R, then the idempotent of R is said to be full.

See also separable algebra and separability idempotent.

Because ab = 0 when neither a nor b is zero and b = 1 a, any non-trivial idempotent an is a zero divisor. This demonstrates that idempotents of this type do not exist in integral domains or division rings. These idempotents are also absent in local rings, but for an alternative reason. When it comes to rings, the only idempotent that can be found in their Jacobson radical is zero[1].

**Contribution to the breakdown**

R idempotents are closely related to R-module decomposition. Given an R-module M and its $End_R(M)$ ring, we get $A \oplus B = M$ if and only if E contains a unique idempotent e such that $A = e(M)$ and $B = (1 - e)(M)$. If and only if 0 and 1 are the only idempotents in E, then M is clearly indecomposable[9].

Whenever M = R, the endomorphism ring $End_R(R) = R$, with each endomorphism arising as the left multiplication of a fixed element of the ring. For the new notation, if and only if there is a unique idempotent e such that$eR = A$ and $(1 - e)R = B$., then $A \oplus B = R$ as right modules. The idempotent generates each direct summand of R.

aRa = Ra is a ring with multiplicative identity if and only if there is a central idempotent. Idempotents determine the direct decompositions of R as a sum of rings, just as idempotents determine the direct decompositions of R as a module.[7] The identity elements of the rings Ri are central idempotents in R, pairwise orthogonal, and their sum is 1. If R is the direct sum of the rings$R_1,...,R_n$, then R is a ring with n members. R is the direct sum of the rings $Ra_1,..,Ra_n$ if and only if the centre idempotents $a_1,..,a_n$ in R are pairwise orthogonal and have a sum of 1. For example, if R has an idempotent central point, then R can be written as the direct sum of the corner rings $aRa$ and $(1 - a)R(1 - a)$. So, if the identity 1 is centrally primitive, and only if it is centrally primitive, then the ring R can't be broken down into smaller rings.

An inductive procedure can be used to try to reduce 1 to a collection of its most fundamental building blocks. It's over if we find that one is centrally primitive. If not, then it is the sum of central orthogonal idempotents, each of which is either primitive or the sum of central idempotents. [11]There's a chance that this process will go on forever, leading to an endless family of central orthogonal idempotents. One form of finiteness criterion for a ring is that it does not have infinite sets of central orthogonal idempotents. There are a number of ways to accomplish this, including requiring a Noetherian ring. If there exists a decomposition $R = c_1R \oplus c_2R \oplus ... \oplus c_nR$ where ci is a centrally primitive idempotent, then R is the direct sum of the corner rings $c_iRc_i$, each of which is a ring irreducible.[8]

## 3. BASIC FACTS

This findings are based on the following, which appears to be on building idempotent elements on a ring using elements from a quotient ring. The major steps in recalling this result for a commutative ring are provided. [14]There is more information in the cited source for the curious reader. Remember that element e is idempotent in ring R if and only if $e^2 = e$, and that elements $e_1$ and $e_2$ are orthogonal in ring R if and only if $e_1 e_2 = 0$.If an idempotent e over R can't be written as the sum of two non-trivial orthogonal idempotent elements, it is said to be primitive.

**Proposition**

Consider the ring R, the nil ideal N of R, and the idempotent element of R/N, f = f + N. Therefore, R contains an idempotent element e such that $\overline{e} = \overline{f}$, where "¯ " indicates the canonical homomorphism from R to R/N. In addition, if R is commutative, then e is a special element.[12]

**Proof**

Since f is idempotent, we get $f^{2-}$ f N, and since N is a nil ideal, we have $(f^2 - f)^n = 0$ for all positive integers n. If $g = 1 - f$, $0 = (fg)^n = f^n g^n$ .

The equation

$1 = 1^{2n-1} = (f + g)^{2n-1} = h + e,$

can be derived from the relationship f + g = 1.

Where

$$h = \sum_{i=o}^{n-1} \binom{2n-1}{i} f^i g^{2n-1-i},$$

$$e = \sum_{i=o}^{n-1} \binom{2n-1}{i} f^i g^{2n-1-i}$$

When $f^n g^n = 0$ and e + h = 1, we get eh = he = 0 To deduce that e f mod N, we need only observe that f 2n1 e mod N and that $f^{2n-1} \equiv e$ mod N and that $f \equiv f^2 \equiv \cdot\cdot\cdot \equiv f^{2n-1}$. It can be shown that e + z is unique if and only if z is nilpotent, so we will use this version of the idempotent element to demonstrate this. $(1 - 2e)z = z^2$is the result of setting$(e + z)^2 = e + z$ . [13]

As a result, $z^3 = (1 - 2e)z^2 = (1 - 2e)^2 z$ through induction

$(1 - 2e)^n z = z^{n+1}$..

This means that z = 0 and e + z = e, since $(1 - 2e)^2 = 1 - 4e + 4e = 1$

## 4. RAISED FORMULAS FOR IDEMPOTENT CALCULATION

### Proposition

Let's say that R is a commutative ring and that N is a nilpotent ideal of index t ≥ 2 in R.

Given that f is an idempotent element in R/N and e is the lifted idempotent element in R corresponding to f, we have[15]

There is a prime integer p such that p t, and for any n in the range [0, N], there is a real number r in the range [0, R] such that

$(e + n)^{p} = e + pnr$

The lifted ideal e is equal to the nilpotency index t of the ideal N if and only if there exists a natural number s > 1 such that sN = 0, and all the prime factors of the number s are greater than or equal to t.

$e = f^{s}$ .

For example, e = f s holds when the nilpotency index of the ideal N equals t = 2 and s ≥ 2, then $e = f^{s}$ .

### Proof

Since n t = 0, and e is idempotent in the ring R, we have the following.[18]

$$(e + n)^{p} = \sum_{j=o}^{p} (\frac{p}{j}) e^{p-j} n^{j}$$

$$= e + \sum_{j=1}^{t-1} (\frac{p}{j}) e n^{j}$$

Since p is prime, it may be divided into equal parts by any integer j such that $\left(\frac{p}{j}\right)$for all $1 \leq j \leq p - 1$..

Moreover, because $t \leq p$,

$$(e + n)^{p} \ + e + pn\,(k_{1}e + k_{2}en + \cdots . k_{t-1}en^{t-2})$$

anywhere

ki =$(\frac{p}{i})/p$.

Therefore,

$(e + n)^{p} = e + pnr,$

By means of

$r = k_1e + k_2en + \cdots . k_{t-1}en^{t-2} \in R.$

Let the primes used to break down s into smaller numbers be$\{p_1p_2 \cdot \cdot \cdot p_m\}$ . We can deduce from (1) that there is some $r_1 \in R$ such that $p1 \geq t$ and $\bar{f} = \bar{e}$, $f = e + n$ for some n N.[16]

$f^{p1} = (e + n)^{p1} = e + p_1nr_1.$

Item 1 also implies that there is a real number $r_2 \in R$ such that

$f^{p1p2} = (f^{p1})_{p2}$

$= (e + p_1nr_1)^{p2}$

$= e + p_2(p_1nr_1)r_2.$

Carrying on with the current procedure There are three real numbers, $r_3, r_4, \cdot \cdot \cdot , r_m \in R$ such that

$f^{s} = e + sn(r_1r_2 \cdot \cdot \cdot r_m).$

In other words,

$f^{s} = e + sh,$

for some N where $h = nr_1r_2 \cdot \cdot \cdot r_s \in N$. Because $h \in N$, and $sN = 0$are both equal to zero, we get $e = f^{s}$.

It is easy to see that all prime factors of s are bigger than or equal to t = 2 if the nilpotency index of the ideal N equals t = 2.

## 5. MUTABLE RINGS WITH A NILPOTENT IDEAL

**Proposition**

Take the commutative ring R and its $k \geq 2$ nilpotent ideal N as an example. Imagine that R/N is a quotient ring with a characteristic $s > 1$. [17] To the extent that f + N is an idempotent component of R/N,

$f^{s^{k-1}}$

is a recursive member of R, the ring of repetitions.

Furthermore, E(R)| = E(R/N)|

**Proof:**

This proposition's proof follows logically. To prove that the collection B = {N, $N^2$ , ..., $N^k$} of ideals of the ring R meets the CNC-condition with the nilpotency index and characteristic of the ideal Ni in the ideal $N^{i+1}$ being $t_i = 2$ and $s_i = s$ for all i = 1, 2, 3, . . . , k – 1

Indeed,

The chain condition is obviously met by the collection B.

Second, $(N^i)\ 2 = N^{2i}$ and i + 1 ≤ 2i for i = 1, 2, 3, . . . , k − 1, This means that B is a nilpotent collection.[19]

For any ring R/N with a characteristic s, there is an integer nN such that

$$\sum_{i=1}^{s} 1_R = n.$$

Then

$sN^i = (1_R + \cdots + 1_R)N^i = nN^i \complement N^{i+1}$

It follows

$sN^i \complement N^{i+1}$ for i = 1, 2, 3, . . . , k − 1.

Also, the collection B meets the characteristic condition because all of the prime parts of $s_i = s$ are greater than or equal to $t_i = 2$. [18]

## 6. CONCLUSIONS

Rings having a nilpotent ideal, commutative group rings RG, where R contains a nilpotent ideal, commutative group rings RG, where R is a chain ring, the group ring ZmG, where Zm is the ring of integers modulo m, and so on all have idempotent members. It transfers idempotent elements from quotient rings to the ambient ring, and under some conditions it provides a simpler formula than the one given in the cited work for determining the lifted idempotent. This conclusion has various ramifications, including the determination of the set of idempotent elements in certain situations, such as when the ring R is a chain ring, contains a nilpotent ideal, or is a commutative ring containing a nilpotent ideal.[20]

## REFERENCES

1. Alahmadi “Decomposition of singular matrices into idempotents”. Linear Multilinear Algebra, Vol 62, issue (1), Page 13–27, 2014.

2. T. Y. Lam and A. Leroy. "Euclidean pairs and quasi-Euclidean rings". Journal of Algebra, Vol 406,issue (1), Page 154–170, 2014.
3. D. de Melo, "On Cyclic and Abelian Codes", IEEE Trans. Inform. Theory, vol.59, Issue (11), Page 7314-7319, 2013.
4. L.H. Kaufmann, "Iterants, Idempotents and Clifford Algebra in Quantum Theory", Vol 2,issue (1), Page 1-36, 2017.
5. Rukhsan Ul Haq "Iterants, Idempotents and Clifford algebra in Quantum Theory",Quantum physics, Vol 2, Issue (11), Page 1-35,2017
6. Apostol Vourdas,"Quantum systems with finite Hilbert space", Reports on Progress in Physics, Vol 67, issue (3),Page 267, 2004.
7. T. Rossmann. "Irreducibility testing of finite nilpotent linear groups", J. Algebra, Vol 324, Issue (5), Page 1114–1124, 2010.
8. T. Rossmann. "Primitivity testing of finite nilpotent linear groups", LMS J. Comput. Math., Vol 14, Issue (2), Page 87–98, 2011.
9. Bak, "Non-abelian K-theory: the nilpotent class of K1 and general stability", K-Theory, Vol 4 ,Issue (2), Page 363 –397,1991.
10. Cox, J. Little and D. O'Shea, "Ideals, varieties and algorithms", Springer-Verlag New York Heidelberg, Undergraduate Texts in Mathematics,Vol 32, Issue (3), Page 150–164,2007.
11. O. Broche and A. del R´ıo, "Wedderburn decomposition of finite group algebras", ´ Finite Fields Appl., vol 13, issue (1), Page 71-79, 2017.
12. J. de la Cruz and W. Willems, "On group codes with complementary duals", Des. Codes Cryptogr., Vol 86, issue (1), Page 2065-2073, 2018.
13. A.M. Dobrotvorskii, "New method of calculating idempotent density matrices and the total energy of many-electron systems.", Theoretical and Experimental Chemistry, vol. 26, Issue (2),Page 1-4 ,1990.
14. R.A. Ferraz and C. Polcino Milies, "Idempotents in group algebras and minimal abelian codes", Finite Fields and Appl., vol 13, issue (1), Page 382-393, 2007
15. F. Ischebeck and R. Rao, "Ideals and Reality, Springer-Verlag Berlin Heidelberg", Springer Monographs in Mathematics, Vol 34, Issue (2), Page 54-65,2009
16. T.Y. Lam, "Serre's Problem on Projective Modules, Springer-Verlag Berlin Heidelberg", Springer Monographs in Mathematics, Vol 32, Issue (4), Page 1–3,2006.
17. S. Lang, "Algebra", Springer-Verlag New York, Graduate Texts in Mathematics, Vol 357, Issue (3), Page 839–850, 2002.
18. Logar and B. Sturmfels, "Algorithms for the Quillen-Suslin Theorem", Journal of Algebra, Vol 145, Issue (2), Page 231–239,1992.
19. J. Rotman, "An Introduction to Homological Algebra, Springer-Verlag New York", Graduate Texts in Mathematics, Vol 11, Issue (3), Page 2–4, 2009.
20. Chen Miaosen, "On products of three triangular matrices over associative rings". — Linear Algebra Applic. Vol 387, issue (1), Page 297–311,2004.